Series expansion of the directed percolation probability

1992 J. Phys. A: Math. Gen. 25 6609

(http://iopscience.iop.org/0305-4470/25/24/014)

J. Phys. A: Math. Gen. 25 (1992) 6609–6615. Printed in the UK

Series expansion of the directed percolation probability

R N Onody and U P C Neves

Departamento de Física e Ciência dos Materiais, Instituto de Física e Química de São Carlos, Universidade de São Paulo, Caixa Postal 369, 13560 São Carlos, São Paulo, Brasil

Received 11 February 1992, in final form 13 July 1992

Abstract. Using a transfer-matrix technique we obtain extended series expansion of the percolation probability for the directed site percolation problem on the square lattice. Our approach reveals a previously unsuspected connection between this problem and the enumeration of the number of ways of dissecting a ball. We show that the method can also be used to determine a series expansion for the mean cluster size. An analysis based on Padé approximants gives estimates of the critical threshold and also of the critical exponent β.

1. Introduction

Percolation theory has a fundamental role in the study of geometric phase transitions. Since it was proposed in its *bond* (Broadbent and Hammersley 1957) and *site* (Domb 1959) versions a huge number of techniques have been used in order to increase our understanding of the subject, e.g. series expansions, Monte Carlo methods, the transfer-matrix technique and various types of calculations spawned by the renormalization group theory (Stauffer 1985).

Directed percolation is an anisotropic variant of percolation in which the lattice is now *oriented*. Acyclic (cyclic) orientation leads to a different (same) universality class from undirected percolation (Blease 1977).

Directed percolation has many possible applications and realizations including crack propagation (Kertész and Vicsek 1980), epidemics with a bias (Grassberger 1985), galactic evolution (Schulman and Seiden 1982) and resistor–diode networks (Redner and Brown 1981). Moreover, it can also be associated to Reggeon field theory (Grassberger and Sundermeyer 1978, Cardy and Sugar 1980), collapse transition for branched polymers (Dhar 1987) and vicious random walkers (Arrowsmith *et al* 1991). Recently a number of results have been obtained for directed percolation regarding, e.g., the fractal dimension at threshold (Hede *et al* 1991), Kasteleyn–Fortuin formulae for a chiral Potts model (Arrowsmith and Essam 1990), a new scaling mechanism for the longitudinal correlation length (Henkel and Privman 1991), conversion site-to-bond method (Duarte 1990), mean-field renormalization group (Neves and Leal da Silva 1991), transfer-matrix methods (ben-Avraham *et al* 1991), series techniques (Onody 1990, Ruskin and Cadilhe 1991) and self-organized criticality (Obukhov 1990).

Focusing our attention on series expansion methods we note that for the pair connectedness moments (low density expansions) very long series are now available (Essam *et al* 1988) for both site and bond percolation on the directed square and triangular lattices. However, for the corresponding series expansion of the percolation probability (high density expansion) the status is not the same and only for the directed *bond* percolation some recent improvements were made (Baxter and Guttmann 1988, Onody 1990). In fact, for the *site* percolation defined on the directed square lattice, the longest previously published series has remained the ten-term series of De'Bell and Essam (1983). In this paper, using a transfer-matrix method, we extend the known series for the percolation probability to 16 terms.

This paper is organized as follows. In section 2 we present the transfer-matrix method we have used to get the series expansion of the percolation probability to order 15. In section 3 we establish a connection between this expansion and the number of dissections of a ball. This connection allows us to increment the series by one more term. We also show how the mean cluster size can be obtained using the same transfer-matrix scheme. A Padé approximant analysis of the series is used in section 4 to estimate the percolation threshold q_c and the critical exponent β. We conclude the paper with a short summary.

2. The transfer-matrix method

Consider a square lattice drawn diagonally as in figure 1. Sites are independently present with probability p and absent with probability $q = 1 - p$ and the edges of the lattice are oriented as shown in the figure 1 (acyclic orientation).

The order parameter for this system is the percolation probability $P(q)$, i.e. the probability that the origin O belongs to an infinite cluster.

As p approaches its critical value p_c from above, $P(p)$ goes to zero according to the power law

$$P(p) \sim (p - p_c)^\beta \tag{2.1}$$

where β is the critical exponent.

For this quantity the longest existent series was evaluated through the enumeration of a lattice animal generating program (De'Bell and Essam 1983). We shall use a transfer-matrix approach that stores information from one row to the next.

Let us denote an occupied site by 1, an empty site by 0 and one configuration (of the 2^N allowed) of the N sites living in the row N by $\{\alpha\}_N = \{\alpha_1, \alpha_2, \ldots, \alpha_N\}$ with α_i being 0 or 1 and corresponding to the state of occupation of the ith site of that row. Let $P(\{\alpha\}_N)$ be the probability of occurrence of such a configuration.

For the second row we have explicitly $P(\{1,1\}) = p^2$, $P(\{1,0\}) = P(\{0,1\}) = pq$ and $P(\{0,0\}) = 0$ (remember that the origin is always occupied and that the

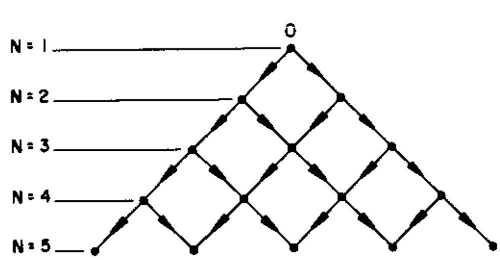

Figure 1. Acyclic orientation of the square lattice. The origin is always occupied and we have labelled the rows as used in the text.

configuration $\{0,0\}$ is forbidden since it interrupts the cluster). We store the four configurations in a 4×3 matrix reserving one line for each configuration and with the columns keeping the coefficients of the corresponding polynomial in the variable q. Summing over all configurations we have

$$P_2(q) = \sum_{\{\alpha\}_2} P(\{\alpha\}_2) = 1 - q^2 \,. \tag{2.2}$$

Now using this information retained up to row 2, we can formulate a procedure that enable us to get *all* configurations probabilities $P(\{\alpha\}_3)$ of row 3. The rules are the following.

(i) Consider a configuration of row 3, say $\{1,0,0\}$.

(ii) Match this with all *compatible* configurations of row 2. By a *compatible* configuration we mean one configuration of the second row that allows *all* the occupied sites of the third row being connected to the origin. In other words, every occupied site of row 3 *must* have at least one occupied nearest neighbour at row 2. This preserves the concept that a site only belongs to the directed cluster if and only if it is connected to the origin.

(iii) For *each* compatible configuration of row 2 multiply the corresponding probability by a factor $p^m q^n$ where m is the number of occupied sites and n is the number of perimeter sites (i.e. *all* empty sites of the row 3 with at least one occupied nearest-neighbour site in row 2) of the third row. Summing over all compatible configurations,

$$P(\{1,0,0\}) = pq^2 P(\{1,1\}) + pq P(\{1,0\}) \,. \tag{2.3}$$

(iv) Doing the same thing for all configurations of the third row, we can then store these probabilities in a matrix $2^3 \times 6$ in the same way that we did with the second row.

It is easy to obtain

$$P_3(q) = \sum_{\{\alpha\}_3} P(\{\alpha\}_3) = 1 - q^2 - 3q^3 + 4q^4 - q^5 \,. \tag{2.4}$$

Proceeding in the same way we can get $P(\{\alpha\}_N)$ and $P_N(q)$ for $N = 4, 5, \ldots$.

We wrote a FORTRAN program to execute the rules above up to row $N = 15$ keeping the polynomials up to order q^{16} (the reason for this will become clear in the next section). It took 36 CPU hours in our CONVEX.

For a finite lattice of N rows we have

$$P_N(q) = \sum_{m=0}^{(N-1)(N+2)/2} a_{Nm} q^m \,. \tag{2.5}$$

For $q < q_c$ we expect that

$$P(q) = \lim_{N\to\infty} P_N(q) \,. \tag{2.6}$$

In fact we can take this as a more precise definition of $P(q)$.

Analysing these polynomials we observe that in going from row N to row $N+1$ leaves the coefficient of $1, q, q^2, \dots, q^N$ *unchanged* such that the percolation probability can be written

$$P(q) = P_\infty = \sum_{N=1}^{\infty} a_{NN} q^N . \tag{2.7}$$

In this way, we have the series expansion

$$P(q) = 1 - q^2 - 3q^3 - 8q^4 - 21q^5 - 56q^6 - 154q^7 - 434q^8 - 1252q^9 - 3675q^{10} - 10\,954q^{11} - 33\,044q^{12} - 100\,676q^{13} - 309\,569q^{14} - 957\,424q^{15} \dots . \tag{2.8}$$

3. Dissections of a ball

Through the analysis of the coefficients a_{Nm} of (2.6) or, more precisely, through analysis of the differences $(a_{N,N+1} - a_{N+1,N+1})$ we have found the following amazing integer sequence: 1, 3, 12, 55, 273, 1428, 7752, 43 263, 246 675, etc. This sequence is exactly an enumeration problem of ball dissections! (See Beineke and Pippert 1971, Sloane 1973.)

The problem of dissections was first formulated by Euler as the number of ways of dissecting a convex polygon of $n+2$ sides into n triangles. This question leads to the Catalan numbers $C_n = (2n)!/n!(n+1)!$ $(n \geqslant 1)$. Such numbers have already appeared in the context of directed *bond* percolation (Baxter and Guttmann 1988). Since the boundary of a polygon is homeomorphic to a circle, such a triangulation can be considered as a dissection of a disc and therefore it is a two-dimensional problem. The three-dimensional analogue corresponds to dissections of a ball.

Let $D_3(n)$ be the number of ways of inserting $n-4$ sheets through a ball having n vertices on its surface in such a way that each sheet contains precisely three of the vertices and so that pairs of sheets meet only on surface curves joining vertices. This problem was solved by Beineke and Pippert (1971):

$$D_3(n) = \frac{(3n-9)!}{(n-3)!(2n-5)!} \tag{3.1}$$

for $n \geqslant 3$.

Turning back to our sequence we can identify

$$a_{N,N+1} - a_{N+1,N+1} = D_3(N+3) = \frac{(3N)!}{(N)!(2N+1)!} . \tag{3.2}$$

That is the reason we have kept the polynomial $P_{15}(q)$ up to order q^{16}. We have $a_{15,16} = 11\,121\,767\,818$ and using (3.2) we can find

$$a_{16,16} = -2\,987\,846 \tag{3.3}$$

incrementing the percolation probability by one more term.

Although we have found the link between the directed site problem and the dissection problem, we don't know its origin. It would be very interesting if one could demonstrate (3.2).

To close this section let us show how the method we have developed so far can also be used to obtain the mean cluster size series $S_N(p)$ (low density series). It is necessary to incorporate, in each row, its site content. Thus, in a row m, we multiply each configuration probability $P(\{\alpha\}_m)$ by the number of *occupied* sites of *this* configuration and sum over *all* possible configurations to get a partial sum $S(m,p)$. Up to row N the mean cluster size will be given by

$$S_N(p) = \sum_{m=1}^{N} S(m,p) \tag{3.4}$$

which gives the correct answer to order p^{N-1} (that is, in going to rows $\geqslant N$ the polynomial coefficients up to order $N-1$ will no longer change).

We can easily calculate by hand the first few partial sums. For example

$$S(1,p) = 1 \qquad S(2,p) = 2p \qquad S(3,p) = -p^3 + 4p^2 \qquad S(4,p) = -4p^4 + 8p^3$$

giving

$$S_4(p) = 1 + 2p + 4p^2 + 7p^3 - 4p^4 \tag{3.5}$$

which it is correct to order p^3.

Although it was amusing to see that our method works equally well for both low and high density series expansion, our results for the mean cluster size do not compete with the previous one based on a Dyson-type equation (Essam *et al* 1988). Indeed they are very different from these as long as we do not go beyond $N = 15$.

4. Analysis of series

Order parameter series are usually well suited to analysis by Dlog Padé approximants. Accordingly, we show in table 1 the standard Dlog Padé approximants for the percolation probabilities series.

Table 1. Dlog Padé approximants to the percolation probability series. Entries to the left (right) are q_c (β) estimates.

N	$[(N-1)/N]$		$[N/N]$		$[(N+1)/N]$	
4	0.293 94	0.2688	0.293 91	0.2687	0.294 27	0.2720
5	0.293 93	0.2688	0.294 33	0.2727	0.294 30	0.2723
6	0.294 27	0.2719	0.294 32	0.2725	0.294 27	0.2720
7	0.294 42	0.2739	0.294 45	0.2745	0.294 48	0.2750
8	0.294 51	0.2759				

Note that there is a general upward trend, and limits around $q_c = 0.2945$ and $\beta = 0.276$ appears entirely attainable. We observe that our estimate for the critical threshold is in disagreement with that recently proposed by Ben-Avraham *et al* (1991) which has a central value at $q_c = 0.293\,478$. From table 1 we see that the approximants exceed that value from the very beginning. Nevertheless our estimated value is completely compatible with $q_c = 0.294\,511 \pm 0.000\,004$ previously proposed by Essam *et al* (1988).

The critical exponent $\beta = 0.276$ is in good agreement with that calculated for the directed *bond* percolation on the square lattice (Baxter and Guttmann 1988) confirming once more that universality holds for both the site and the bond formulations of the percolation problem.

As our values for the critical threshold and critical exponent are less precise than the values obtained by Essam *et al* (1988) and Baxter and Guttmann (1988) we can use their values in order to get refinements.

Following Gaunt and Guttmann (1974) we wrote down Padé approximants to the series

$$(q - q_c)\frac{\mathrm{d}\ \ln P(q)}{\mathrm{d}q} \tag{4.1}$$

and the results are shown in table 2.

Table 2. Padé approximants to the series (4.1) using $q_c = 0.294\,511$.

N	$[(N-1)/N]$	$[N/N]$	$[(N+1)/N$
4	0.272 929	0.260 376	0.275 017
5	0.274 829	0.275 265	0.274 598
6	0.279 587	0.275 471	0.275 643
7	0.275 610	0.275 790	0.275 789
8	0.275 789		

Now, conversely, we can use a good estimate of β to form Padé approximants to the series

$$[P(q)]^{1/\beta} \tag{4.2}$$

to get better estimates of q_c. The results are given in table 3.

Table 3. Padé approximants to the series (4.2) using $\beta = 0.2764$.

N	$[(N-1)/N]$	$[N/N]$	$[(N+1)/N$
5	0.294 682	0.294 661	0.294 143
6	0.294 710	0.294 461	0.293 865
7	0.294 643	0.294 562	0.294 548
8	0.294 545	0.294 541	

5. Summary

We have calculated an exact series expansion for the percolation probability of the directed site percolation problem on the square lattice. This expansion increases the known series from order q^{10} to order q^{16}. The method used was a transfer-matrix method which simultaneously allowed us to establish an interesting connection between this problem and the combinatorial problem of enumerating the number of dissections of a ball. Our approach also permits us to extract series for the mean cluster size. Finally, we have given estimates for the critical threshold and the critical exponent of the order parameter through an analysis based on Padé approximants.

References

Arrowsmith D K and Essam J W 1990 *Phys. Rev. Lett.* **65** 3068

Arrowsmith D K, Mason P and Essam J W 1991 *Physica* **177** 267

Baxter R J and Guttmann A J 1988 *J. Phys. A: Math. Gen.* **21** 3193

Beineke L W and Pippert R E 1971 *Math. Ann.* **191** 87

Ben-Avraham D, Bidaux R and Schulman L S 1991 *Phys. Rev.* A **43** 7093

Blease J 1977 *J. Phys. C: Solid State Phys.* **10** 917

Broadbent S R and Hammersley J M 1957 *Proc. Camb. Phil. Soc.* **53** 629

Cardy J L and Sugar R L 1980 *J. Phys. A: Math. Gen.* **13** L423

De'Bell K and Essam J W 1983 *J. Phys. A: Math. Gen.* **16** 3145

Dhar D 1987 *J. Phys. A: Math. Gen.* **20** L847

Domb C 1959 *Nature* **184** 509

Duarte J A M S 1990 *Z. Phys.* B **80** 299

Essam J W, Guttmann A J and De'Bell K 1988 *J. Phys. A: Math. Gen.* **21** 3815

Gaunt D S and Guttmann A J 1974 *Phase Transitions and Critical Phenomena* vol 3, ed C Domb and M S Green (New York: Academic)

Grassberger P 1985 *J. Phys. A: Math. Gen.* **18** L215

Grassberger P and Sundermeyer K 1978 *Phys. Lett.* **77B** 220

Hede B, Kertész J and Vicsek T 1991 *J. Stat. Phys.* **64** 829

Henkel M and Privman V 1991 *Mod. Phys. Lett.* B **5** 555

Kertész J and Vicsek T 1980 *J. Phys. C: Solid State Phys.* **13** L343

Neves A and Leal da Silva J K 1991 *J. Phys. A: Math. Gen.* **24** 5363

Obukhov S P 1990 *Phys. Rev. Lett.* **65** 1395

Onody R N 1990 *J. Phys. A: Math. Gen.* **23** L335

Redner S and Brown A C 1981 *J. Phys. A: Math. Gen.* **14** L285

Ruskin H J and Cadilhe A 1991 *J. Non-Cryst. Solids* **127** 114

Schulman L S and Seiden P E 1982 *J. Stat. Phys.* **27** 83

Sloane N J A 1973 *A Handbook of Integer Sequences* (New York: Academic)

Stauffer D 1985 *Introduction to Percolation Theory* (London: Taylor and Francis)